MTH 261 Midterm Exam Sample Key

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Name: ______________________________

For instructor use only

Score : /125

Some notes from Damien about the exam:

- You will have 2 hours 20 minutes to complete this exam. When you are finished, please turn it in to Damien.
- You may use a graphing calculator, but once you begin the exam, you *may not* use an internet sources or any other external source for assistance. This includes your notes, peers, textbook, tutors, other human beings, robots, or any other source I may have failed to mention.
- You may ask Damien for clarification on any problems, but you *may not* converse with anyone else regarding this exam.
- If you have any questions, ask Damien!
- The use of scratch paper is encouraged and should be included at the end of the exam. Please clearly label all work for each prompt.
- Please read all of the instructions carefully.
- Please show all of your work to each prompt. This is exceedingly important!
- Please use exact values unless otherwise stated.
- Please use vector notation any time you are writing a vector.
- Please show all row reduction steps.
- May the Force be with you!

Furthermore, please read all of the conditions below before beginning the exam. Indicate that you have read and understand these conditions by placing your initials on the line below.

- I have and will follow all of the guidelines stated above.
- I have not and will not cheat or violate any part of the Academic Integrity Policies.
- I will not provide or distribute any portion of this document to anyone other than the instructor.
- All work attached herein is my own authentic work.

Initials : ______________

(8) 1. Suppose A, B, and C are matrices with the following sizes: A is $42 \times \alpha$, B is $37 \times \beta$, and C is $\gamma \times \delta$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{N}$.

(a) Is it possible that $C = BA$? If so, then what must β be? If not, please explain why as specifically as you can.

Solution: Yes, it is. In this case, $\beta = 42$.

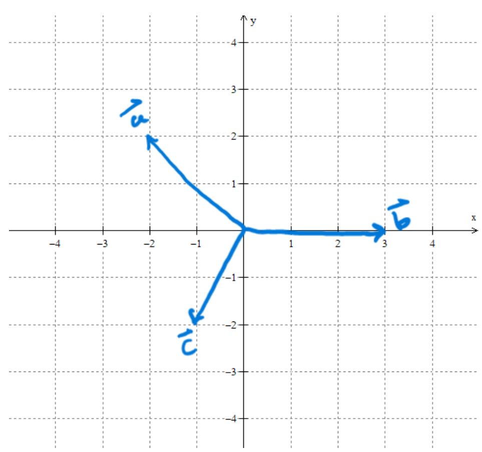
(b) Is it possible that $C = B + A$? If so, then what must β be? If not, please explain why as specifically as you can.

Solution: No, this is not possible. The number of rows of A and the number of rows of B do not match.

(9) 2. Let $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$. On the provided set of axes, graph the following vectors as arrow vectors beginning at $\mathbf{0}$. Clearly label each vector on the graph.

(a) $\mathbf{a} = -2\mathbf{u}$ (b) $\mathbf{b} = \mathbf{u} + \mathbf{v}$ (c) $\mathbf{c} = \mathbf{u} - \mathbf{v}$

Solution:



(8) 3. Determine if the statement is True or False. If the statement is True, you need only write "True" and do not need to provide a justification (though one may provide partial credit). If the statement is False, write "False" and justify your conclusion as specifically as possible. (Do not write "T" or "F"; please write the full word)

(a) The matrix equation $A\mathbf{x} = \mathbf{b}$ always has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Solution: False. The matrix A is not necessarily invertible.

(b) If A, B, C are $n \times n$ matrices, then $(ABC)^T = A^T B^T C^T$.

Solution: False. $(ABC)^T = C^T B^T A^T$.

(10) 4. Solve the system of equations using Gaussian Elimination. Provide your conclusion in parametric vector form.

$$\begin{cases} x_1 + 2x_2 = 1 \\ x_1 + 2x_2 - x_3 = 2 \\ 2x_1 + 4x_2 + 2x_3 = 0 \end{cases}$$

Solution: Begin by converting the system into an augmented matrix and row-reducing.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & -1 & 2 \\ 2 & 4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 2 & 4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This last matrix produces the system

$$\begin{cases} x_1 + 2x_2 = 1 \\ x_3 = -1 \\ 0 = 0 \end{cases}$$

So $x_1 = 1 - 2x_2$, $x_2 = x_2$, and $x_3 = -1$. This means

$$\begin{aligned} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 1 - 2x_2 \\ x_2 \\ -1 \end{bmatrix} \\ &= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \end{aligned}$$

(8) 5. Let $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, where

$$\mathbf{u} = \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} \quad , \quad \mathbf{v} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \quad , \quad \mathbf{w} = \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}$$

Determine if S is linearly independent or not. If the set is linearly dependent, then find a linear dependence relation. If the set is linearly independent, justify your conclusion as specifically as possible.

Solution: We need to solve $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$. We augment and row reduce.

$$\begin{bmatrix} 2 & -1 & 1 & 0 \\ 7 & 4 & 6 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 7 & 4 & 6 & 0 \\ 2 & -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 2 & -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & -3 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This tells us that $x_1 = \frac{-2}{3}x_3$, $x_2 = \frac{-1}{3}x_3$, and x_3 is free. If we choose $x_3 = 3$, then we have $(x_1, x_2, x_3) = (-2, -1, 3)$. Plugging this into our homogeneous equation gives us our linear dependence relation.

$$-2\mathbf{u} - \mathbf{v} + 3\mathbf{w} = \mathbf{0}$$

6. Let $\mathbf{u}, \mathbf{v}, \mathbf{b} \in \mathbb{R}^3$ such that

$$\mathbf{u} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \quad , \quad \mathbf{v} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ -6 \end{bmatrix}$$

(5) (a) Determine if $\mathbf{b} \in \text{span}\{\mathbf{u}, \mathbf{v}\}$ or not. Show all work that leads to your conclusion.

Solution: The vector $\mathbf{b} \in \text{span}\{\mathbf{u}, \mathbf{v}\}$ iff $\mathbf{b} = x_1\mathbf{u} + x_2\mathbf{v}$ is consistent. We will solve this equation by first setting up an augmented matrix and row-reducing.

$$\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{b} \end{bmatrix} \sim \begin{bmatrix} -3 & 1 & 10 \\ 1 & 5 & 3 \\ 2 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 \\ -3 & 1 & 10 \\ 2 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 16 & 19 \\ 2 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 16 & 19 \\ 0 & -10 & -12 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & \frac{19}{16} \\ 0 & 1 & \frac{6}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & \frac{19}{16} \\ 0 & 0 & \frac{-17}{48} \end{bmatrix}$$

Because the last column of REF $\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{b} \end{bmatrix}$ is a pivot column, the original equation must be inconsistent. It follows that $\mathbf{b} \notin \text{span}\{\mathbf{u}, \mathbf{v}\}$.

(5) (b) Determine if $\{\mathbf{u}, \mathbf{v}, \mathbf{b}\}$ is linearly independent or not. If the set is linearly dependent, then find a linear dependence relation. If the set is linearly independent, justify your conclusion as specifically as possible.

Solution: From part (a), we know that $\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{b} \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 3 \\ 0 & 1 & \frac{19}{16} \\ 0 & 0 & \frac{-17}{48} \end{bmatrix} \sim I$. By the Invertible Matrix Theorem, we can conclude that $\{\mathbf{u}, \mathbf{v}, \mathbf{b}\}$ is a linearly independent set.

(15) 7. Evaluate the matrix expression or explain why the expression is undefined. Show all of your work to support your conclusion.

(a) $3\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}^T - \begin{bmatrix}1 & 1 & 1\\ 2 & -1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 1\\ 1 & 0\end{bmatrix}$

Solution:

$$\begin{aligned} 3\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}^T - \begin{bmatrix}1 & 1 & 1\\ 2 & -1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 1\\ 1 & 0\end{bmatrix} &= 3\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix} - \begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}\\ &= \begin{bmatrix}3 & 4\\ 2 & -4\end{bmatrix}\end{aligned}$$

(b) $2I_4 + \begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix}$

Solution: Notice that the product of matrices will produce a 2×2 matrix while $2I_4$ is a 4×4 matrix. We cannot add matrices of different sizes, so this expression is undefined.

(c) $\begin{bmatrix}0 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}3 & 1 & 4\\ 1 & 5 & 9\\ 2 & 6 & 5\\ 3 & 5 & 8\\ 9 & 7 & 9\end{bmatrix}$

Solution: The first matrix is an elementary matrix that will interchange rows 1 and 4. Thus, we get

$$\begin{bmatrix}3 & 5 & 8\\ 1 & 5 & 9\\ 2 & 6 & 5\\ 3 & 1 & 4\\ 9 & 7 & 9\end{bmatrix}$$

8. The following are Fibonacci matrices. Determine if A is invertible or singular. If A is invertible, find A^{-1}. If A is singular, justify your conclusion as specifically as possible.

(5) (a) $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

> **Solution:** Let's begin by computing $\det A = (1)(3) - (1)(2) = 1$. Thus, A is invertible since $\det A \neq 0$. Then $A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$.

(6) (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 5 & 8 \end{bmatrix}$

> **Solution:** A is invertible if $A \sim I$. Let's row-reduce $\begin{bmatrix} A & I \end{bmatrix}$.
>
> $$\begin{aligned} \begin{bmatrix} A & I \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 3 & 5 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 3 & 5 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 5 & 8 & -3 & 0 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 5 & 8 & -3 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 8 & \frac{1}{2} & \frac{-5}{2} & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{16} & \frac{-5}{16} & \frac{1}{8} \end{bmatrix} \end{aligned}$$
>
> It follows that $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{1}{16} & \frac{-5}{16} & \frac{1}{8} \end{bmatrix}$.

(8) 9. Provide the following definitions as specifically as possible.

(a) Define what it means for a vector $\mathbf{u}$ to be in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \in \mathbb{R}^n$.

> **Solution:** A vector $\mathbf{u}$ is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ if $\mathbf{u}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.

(b) Define what it means for the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ to be linearly independent.

> **Solution:** An ordered set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent if the homogeneous equation
>
> $$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$
>
> has only the trivial solution.

10. Let $A = \begin{bmatrix} 7 & -4 & 32 & 24 \\ -3 & 1 & -13 & -11 \\ 8 & -1 & 33 & 31 \end{bmatrix}$. Use the fact that $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 4 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ to answer the following questions.

(6) (a) Find the general solution to $A\mathbf{x} = \mathbf{0}$, and give your answer in parametric **vector** form.

Solution: Since $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 4 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we know $\text{RREF}(\begin{bmatrix} A & \mathbf{0} \end{bmatrix}) = \begin{bmatrix} 1 & 0 & 4 & 4 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.
We can extract the equations $x_1 + 4x_3 + 4x_4 = 0$ and $x_2 - x_3 + x_4 = 0$. Thus, we have a solution of

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4x_3 - 4x_4 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(4) (b) Do the columns of A span $\mathbb{R}^3$? Why or why not? Justify your conclusion as specifically as possible.

Solution: Since RREF A has only two pivots, we can determine that the span of the columns of A is a plane in $\mathbb{R}^3$. This tells us that the columns of A do not span $\mathbb{R}^3$.

(12) 11. A system of equations in the variables x_1 and x_2 is given below, where $\alpha, \beta \in \mathbb{R}$. Determine what values α and β must be that will produce the desired number of solutions in each part.

$$\begin{aligned} x_1 + 3x_2 &= \alpha \\ 4x_1 - \beta x_2 &= 8 \end{aligned}$$

(a) No solutions. (b) Exactly one solution. (c) Infinitely many solution.

Solution: Let's produce the augmented matrix for the system and reduce it a bit.

$$\begin{bmatrix} 1 & 3 & \alpha \\ 4 & -\beta & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & \alpha \\ 0 & -12-\beta & 8-4\alpha \end{bmatrix}$$

- The system is inconsistent (no solutions) if the last row has the form $\begin{bmatrix} 0 & 0 & \blacksquare \end{bmatrix}$ where $\blacksquare \neq 0$. This happens when $\beta = -12$ and $\alpha \neq 2$.
- The system has exactly one solution if the last row has the form $\begin{bmatrix} 0 & \blacksquare & * \end{bmatrix}$, where $\blacksquare \neq 0$ and $* \in \mathbb{R}$. This happens when $\beta \neq -12$ and $\alpha \in \mathbb{R}$.
- The system has infinitely many solutions when there is a free variable. Since the first column is a pivot column, x_1 is a basic variable. The variable x_2 is free if the second and third columns are not pivot columns. This will happen if the last row is all 0s, and this happens when $\beta = -12$ and $\alpha = 2$.

(6) 12. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first reflects points across the line $x_1 = x_2$ and then rotates them about the origin $\frac{\pi}{2}$ radians. Find the standard matrix for T.

Solution: The standard matrix A for the linear transformation T will be the 2×2 matrix whose columns are $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$.

$$\mathbf{e}_1 \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$\mathbf{e}_2 \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

It follows that $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

13. Define the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + x_2$.

(4) (a) Evaluate $T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$.

Solution:

$$T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right) = (u_1 + v_1) + (u_2 + v_2)$$
$$T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = (u_1 + u_2) + (v_1 + v_2)$$

(4) (b) Evaluate $T\left(c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)$ and $cT\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)$.

Solution:

$$T\left(c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = cu_1 + cu_2$$
$$cT\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = c(u_1 + u_2) = cu_1 + cu_2$$

(2) (c) Is T a linear transformation? Circle one: $\boxed{\text{Yes}}$ No

Your work above in parts (a) and (b) serve as justification, so no additional justification is necessary.